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A closed form solution about stress intensity factors of shear modes for 3-D finite bodies with eccentric cracks by the energy release rate method

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Abstract

In this paper, a new analytical-engineering method of closed form solution about stress intensity factors of shear modes for 3-D finite bodies with eccentric cracks is derived by means of the energy release rate method and relevant given 2-D stress intensity factors. This method is both accurate and efficient. Hence a complete series about useful new results of stress factors K_{II} and K_{III} can be obtained.

Keywords: Closed form solution; Stress intensity factor; Shear mode; Eccentric crack; 3-D finite body

1. Introduction

The 3-D stress intensity factors are very important controlling parameters in linear elastic fracture mechanics. Exact solutions of the 3-D stress intensity factors can only be obtained for infinite bodies with embedded cracks. There are several approximate methods of solution for finite bodies, but unfortunately, all of them are very time-consuming. Therefore, there are very few results about 3-D eccentric crack problems, especially for the case of shear modes.

In Wang et al. (1990a, b), new analytical-engineering methods to obtain the closed form solution for stress intensity factors of mode I about eccentric cracks, and closed form solution for stress intensity factors of mode II and III about non-eccentric cracks were advanced, respectively. In this paper, the theory is extended, and a closed form solution about stress intensity factors of shear modes for 3-D finite bodies with eccentric cracks is derived by means of energy release rate method.

The procedure of solution can be summarized as follows:

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(1) To determine the 2-D crack sliding and crack tearing displacements with given 2-D stress intensity factors K_{II} and K_{III}, respectively, by means of energy release rate method.
(2) To establish the modes of 3-D crack surface displacement by given 2-D ones, with condition of displacement compatibility.
(3) To establish the relationship between 3-D stress intensity factors (K_{II} and K_{III}) and generalized 3-D crack surface displacements by means of near field stress and displacement expressions.
(4) To determine the generalized 3-D crack surface displacement and the 3-D stress intensity factors with energy release rate method.

For the convenience of understanding, the third and fourth steps are discussed at first, and then the first and second ones.

2. 3-D stress intensity factors and crack surface displacement

Figure 1 shows a 3-D cracked body subjected to shear load. Two kinds of sections would be introduced: transversal sections parallel to the x–y plane and longitudinal sections parallel to y–z plane as shown in Fig. 2. It can be assumed that the crack surface displacement is along the same direction with shear load acting on crack surfaces perpendicular to the y-axis and can be expressed into the following pattern

$$w(x,z) = w_{01}h_1(x,z) + w_{02}h_2(x,z) \tag{1}$$

$$w^2(x,z) = w_{0i}w_{0j}H_{ij}(x,z) \tag{2}$$

where, $h_1(x,z)$ and $h_2(x,z)$ are symmetric and anti-symmetric distribution functions of crack surface displacements corresponding to anti-symmetric and symmetric crack surface shear displacements,

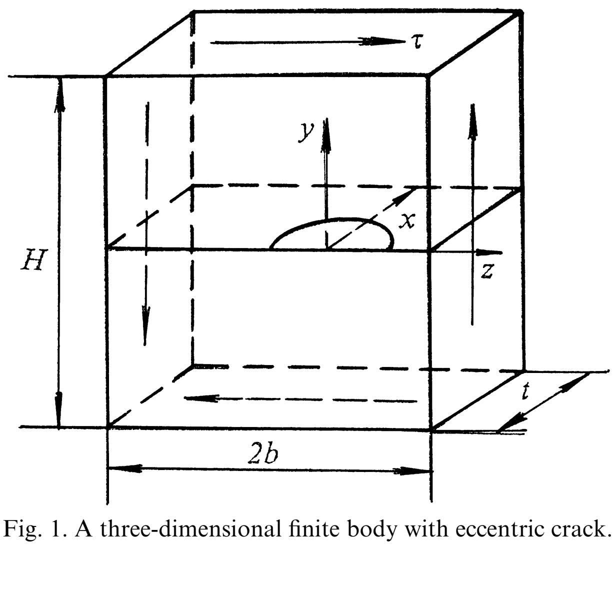

Fig. 1. A three-dimensional finite body with eccentric crack.

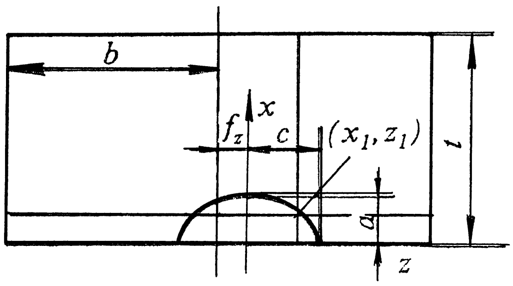

Fig. 2. The cross-section containing a eccentric elliptical crack.

respectively, w_{01} and w_{02} are the corresponding generalized crack surface displacements (amplitudes) corresponding to $h_1(x,z)$ and $h_2(x,z)$, respectively. Furthermore,

$$H_{ij}(x,z) = h_i(x,z)h_j(x,z) \tag{3}$$

where, $H_{ij}(x,z)$ are products of distribution functions and which equals to zero along the crack front and has second derivatives in the vicinity of the crack front.

It must be emphasized that H_{ij} is a symmetric function when $i = j$, and H_{ij} is an antisymmetric function when $i \neq j$.

If $P(x_1, z_1)$ is an arbitrary point on the crack front, then in the vicinity of point P, the square of crack surface displacement $w^2(x,z)$ can be expanded into Taylor's series, and the higher terms can be neglected. So, we have

$$\begin{aligned} w^2(x,z) &= w_{0i}w_{0j}\left\{\frac{\partial H_{ij}}{\partial x}\bigg|_{(x_1,z_1)}(x-x_1)+\frac{\partial H_{ij}}{\partial z}\bigg|_{(x_1z_1)}(z-z_1)\right\} \\ &= w^2(x,z_1)+w^2(x_1,z) \end{aligned} \tag{4}$$

$$w^2(x_1,z) = w_{0i}w_{0j}\left\{\frac{\partial H_{ij}}{\partial z}\bigg|_{(x_1,z_1)}(z-z_1)\right\} \tag{5}$$

$$w^2(x,z_1) = w_{0i}w_{0j}\left\{\frac{\partial H_{ij}}{\partial x}\bigg|_{(x_1,z_1)}(x-x_1)\right\} \tag{6}$$

where, $w(x,z_1)$ is the anti-plane crack tearing displacement (CTD) of the transversal section, and $w(x_1,z)$ is the in-plane crack sliding displacement (CSD) of the longitudinal section passing through the same point P. Equation (4) can be called the Pythagorean theorem of the crack surface displacement.

As shown in Fig. 3, a normal slice can be introduced, and the crack surface displacement can be separated into two components: the in-plane CSD $w\cdot\cos\psi$ and the anti-plane CTD $w\cdot\sin\psi$. In the vicinity of an arbitrary point $P(x_1,z_1)$ on the crack front, the two components can be expressed by means of corresponding stress intensity factors as follows:

$$w^2\cos^2\psi = \frac{8K_{\mathrm{II}}^2}{\pi E_n^2}r, \quad w^2\sin^2\psi = \frac{2K_{\mathrm{III}}^2}{\pi\mu^2}r \tag{7}$$

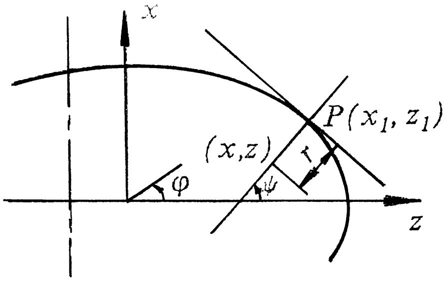

Fig. 3. The crack surface displacement of a normal slice separated into two components.

$$w\cos\psi = \sqrt{\frac{8}{\pi}\frac{K_{\mathrm{II}}}{E_n}}\sqrt{r}, \quad w\sin\psi = \sqrt{\frac{2}{\pi}\frac{K_{\mathrm{III}}}{\mu}}\sqrt{r} \tag{8}$$

where, E_n is the generalized Young's modulus of the normal slices and can be expressed by following equality

$$E_n = E + (E_1 - E)f(\psi); \quad E_1 = E/(1-v^2);\ f(\psi) \in [0, 1] \tag{9}$$

v is Poisson's ratio, and μ is the shear modulus

$$\mu = \frac{E}{2(1+v)} \tag{10}$$

For a plate with an embedded fully elliptical crack, every normal slice can be assumed in a state of plane strain, then we have $f(\psi) = 1$. For a plate with a surface semi-elliptical crack, the stress state of the normal slice varies from plane stress at $\psi = 0$ to plane strain at $\psi = \pi/2$. So it can be assumed that $f(\psi) = \sin\psi$.

From eqns (7) and (4), it can be obtained that

$$\begin{aligned} K_{\mathrm{II}}^2 &= -\frac{\pi}{8}E_n^2 w_{0i}w_{0j}\left\{\frac{\partial H_{ij}}{\partial x}\sin\psi + \frac{2H_{ij}}{\partial z}\cos\psi\right\}_p \cos^2\psi \\ &= \frac{\pi}{8}E_n^2\, w_{0i}w_{0j}\lim_{r\to 0}\frac{1}{r}[H_{ij}(x, z_1) + H_{ij}(x_1, z)]\cos^2\psi \end{aligned} \tag{11}$$

$$\begin{aligned} K_{\mathrm{III}}^2 &= -\frac{\pi}{2}\mu^2 w_{0i}w_{oj}\left\{\frac{\partial H_{ij}}{\partial x}\sin\psi + \frac{\partial H_{ij}}{\partial z}\cos\psi\right\}_p \sin^2\psi \\ &= \frac{\pi}{2}\mu^2 w_{0i}w_{0j}\lim_{r\to 0}\frac{1}{r}[H_{ij}(x, z_1) + H_{ij}(x_1, z)]\sin^2\psi \end{aligned} \tag{12}$$

So, K_{II} and K_{III} can be obtained, if amplitudes w_{0i} and products of distribution functions H_{ij} are determined.

3. Basic differential equation

To determine the displacement amplitudes w_{0i}, the increment of potential energy $d\Pi$ during crack growth should be studied. According to the principle of superposition, the load acting on the boundary of the body can be transferred onto the surface of the crack and can be expressed as $\tau = \tau_0 s(x, z)$, where τ_0 is the generalized force, and $s(x, z)$ is the load distribution function. The potential energy of a linear elastic cracked body will be

$$\Pi = -\int_A \tau w \, dA = -\tau_0 \pi ac\{w_{01} B_1 + w_{02} B_2\}$$
$$= -\tau_0 \pi ac w_{01} B \tag{13}$$

where, A is the crack area and

$$B_1 = \frac{1}{\pi ac} \int_A s(x, z) \cdot h_1(x, z) \, dA$$
$$B_2 = \frac{1}{\pi ac} \int_A s(x, z) \cdot h_2(x, z) \, dA$$
$$B = B_1 + w_{02} B_2 / w_{01} \tag{14}$$

The crack front is assumed to be elliptical with semi-axis a and c parallel to x and z axes, respectively. To determine w_{01} and w_{02}, two kinds of virtual crack extension are considered, and the increment of potential energy $d\Pi$ during crack growth should be studied.

(a) Proportional extension.

The virtual proportional extension of the crack can be expressed by

$$da = a \, dg_1, \quad dc = c \, dg_1 \tag{15}$$

as shown in Fig. 4(a).

Now, G is used to represent the energy release rate of an arbitrary normal slice with unit thickness along crack front. It is well known that

$$G = \frac{1}{E_n} K_{\mathrm{II}}^2 + \frac{1}{2\mu} K_{\mathrm{III}}^2 \tag{16}$$

Then, the energy release of the three-dimensional cracked body is

$$d\Pi = -\int_s G \, dr \, ds \tag{17}$$

where, s is the crack front, dr and ds are used to represent the amount of crack extension and the thickness of the normal slice, respectively. Then

$$ds = \sqrt{c^2 \sin^2 \varphi + a^2 \cos^2 \varphi} \, d\varphi, \quad dr = (da) \sin \varphi \sin \psi + (dc) \cos \varphi \cos \psi \tag{18}$$

$$\sin\psi = \frac{c\sin\varphi}{\sqrt{a^2\cos^2\varphi + c^2\sin^2\varphi}}, \quad \cos\psi = \frac{a\cos\varphi}{\sqrt{a^2\cos^2\varphi + c^2\sin^2\varphi}} \tag{19}$$

In the above two equations, φ is the parametric angle of the elliptical crack front.

So, after substituting eqns (11), (12), (16), (18) and (19) into eqn (17), with consideration of the symmetric and antisymmetric characters of H_{ii} and H_{ij} $(i \neq j)$, $\mathrm{d}\Pi$ will be transformed into following pattern

$$\begin{aligned} \mathrm{d}\Pi &= -\int_s G\,\mathrm{d}r\,\mathrm{d}s \\ &= -\sum_{i=1}^{2}\frac{\pi}{8}E_1 w_{0i}^2 ac\left\{\frac{I_i^*}{a} + \frac{J_i^*}{c}\right\}\mathrm{d}g - \sum_{i=1}^{2}\frac{\pi}{4}\mu w_{0i}^2 ac\left\{\frac{I_i^{**}}{a} + \frac{J_i^{**}}{c}\right\}\mathrm{d}g \end{aligned} \tag{20}$$

where

$$\begin{aligned} I_i^* &= a\int_s \frac{E_n}{E_I}\lim_{r\to 0}\frac{1}{r}H_{ii}(x,z_1)\cos^2\psi\,\mathrm{d}\varphi \\ &= -a\int_s \frac{E_n}{E_I}\left.\frac{\partial H_{ii}}{\partial x}\right|_{(x_1,z_1)}\sin\psi\cos^2\psi\,\mathrm{d}\varphi \\ I_i^{**} &= a\int_s \lim_{r\to 0}\frac{1}{r}H_{ii}(x,z_1)\sin^2\psi\,\mathrm{d}\varphi \\ &= -a\int_s \left.\frac{\partial H_{ii}}{\partial x}\right|_{(x_1,z_1)}\sin^3\psi\,\mathrm{d}\varphi \\ J_i^* &= c\int_s \frac{E_n}{E_I}\lim_{r\to 0}\frac{1}{r}H_{ii}(x_1,z)\cos^2\psi\,\mathrm{d}\varphi \\ &= -c\int_s \frac{E_n}{E_I}\left.\frac{\partial H_{ii}}{\partial z}\right|_{(x_1,z_1)}\cos^3\psi\,\mathrm{d}\varphi \\ J_i^{**} &= c\int_s \lim_{r\to 0}\frac{1}{r}H_{ii}(x_1,z)\sin^2\psi\,\mathrm{d}\varphi \\ &= -c\int_s \left.\frac{\partial H_{ii}}{\partial z}\right|_{(x_1,z_1)}\cos\psi\sin^2\psi\,\mathrm{d}\varphi \end{aligned} \tag{21}$$

In the above equations, repetitions of subscripts do not mean summation with respect to it.

Taking the differential of (13) and comparing it with (20), the first differential equation about w_{01} and w_{02} can be obtained.

$$\frac{\partial w_{01}}{\partial g_1} + \left(2 + \frac{1}{B}\frac{\partial B}{\partial g_1}\right)w_{01} = \sum_{i=1}^{2}\left[\frac{E_1}{8\tau_0 B}\left(\frac{I_i^*}{a} + \frac{J_i^*}{c}\right) + \frac{\mu}{4\tau_0 B}\left(\frac{I\ _i^*}{a} + \frac{J_i^{**}}{c}\right)\right]w_{0i}^2 \tag{22}$$

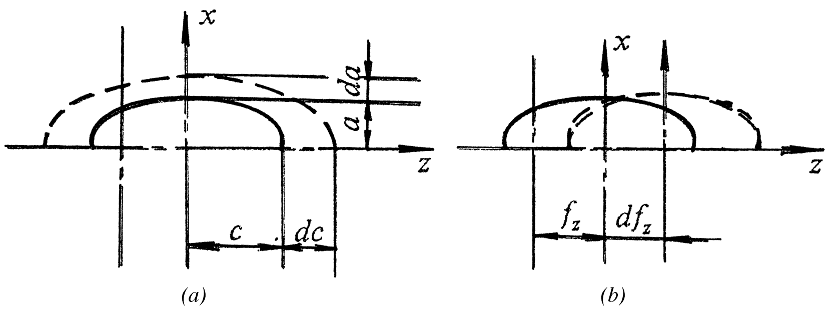

Fig. 4. Two kinds of virtual crack extensions.

(b) Rigid translation along the z axis.

The virtual rigid translation of the crack along the z axis can be expressed by

$$\mathrm{d}a = \mathrm{d}c = 0; \quad \mathrm{d}f_z = f_z\, \mathrm{d}g_2 \tag{23}$$

as shown in Fig. 4(b).

With the same procedure as (a), the second differential equation about w_{01} and w_{02} can be established

$$\frac{\partial w_{01}}{\partial g_2} + \left(\frac{1}{B}\frac{\partial B}{\partial g_2}\right) w_{01} = \left[\frac{E_1}{4\tau_0 B}\left(\frac{M^*}{a} + \frac{N^*}{c}\right) + \frac{\mu}{2\tau_0 B}\left(\frac{M^{**}}{a} + \frac{N^{**}}{c}\right)\right] w_{01} w_{02} \frac{f_z}{c} \tag{24}$$

where,

$$M^* = a\int_s \frac{E_n}{E_1}\lim_{r\to 0}\frac{1}{r} H_{12}(x, z_1)\cos\varphi\cos^2\psi\, \mathrm{d}\varphi$$

$$= -a\int_s \frac{E_n}{E_1}\frac{\partial H_{12}}{\partial x}\bigg|_{(x_1,z_1)} \sin\psi\cos\varphi\cos^2\psi\, \mathrm{d}\varphi$$

$$M^{**} = a\int_s \lim_{r\to 0}\frac{1}{r} H_{12}(x, z_1)\cos\varphi\sin^2\psi\, \mathrm{d}\varphi$$

$$= -a\int_s \frac{\partial H_{12}}{\partial x}\bigg|_{(x_1,z_1)} \sin^3\psi\cos\varphi\, \mathrm{d}\varphi$$

$$N^* = c\int_s \frac{E_n}{E_1}\lim_{r\to 0}\frac{1}{r} H_{12}(x_1, z)\cos\varphi\cos^2\psi\, \mathrm{d}\varphi$$

$$= -c\int_s \frac{E_n}{E_1}\frac{\partial H_{12}}{\partial z}\bigg|_{(x_1,z_1)} \cos^3\psi\cos \mathrm{d}\varphi\, \mathrm{d}\varphi$$

$$N^{**} = c\int_s \lim_{r\to 0}\frac{1}{r}H_{12}(x_1,z)\cos\varphi\sin^2\psi\,\mathrm{d}\varphi$$
$$= -c\int_s \left.\frac{\partial H_{12}}{\partial z}\right|_{(x_1,z_1)}\cos\psi\cos\varphi\sin^2\psi\,\mathrm{d}\varphi \quad (25)$$

Now, there are two non-linear differential equations (22) and (24) of first-order about w_{01} and w_{02}, and the closed form solutions of them are to be established.

4. Closed form solution

Let

$$\zeta = w_{02}/w_{01}$$
$$I^* = I_1^* + \zeta^2 I_2^* \quad J^* = J_1^* + \zeta^2 J_2^*$$
$$I^{**} = I_1^{**} + \zeta^2 I_2^{**} \quad J^{**} = J_1^{**} + \zeta^2 J_2^{**} \quad (26)$$

After substitution, eqn (22) will be transformed into the following

$$\frac{\partial w_{01}}{\partial g_1} + \left(2 + \frac{1}{B}\frac{\partial B}{\partial g_1}\right)w_{01} = \left[\frac{E_1}{8\tau_0 B}\left(\frac{J^*}{a} + \frac{J^*}{c}\right) + \frac{\mu}{4\tau_0 B}\left(\frac{I^{**}}{a} + \frac{J^{**}}{c}\right)\right]w_{01}^2 \quad (27)$$

To obtain the closed form solution of w_{01}, two extreme cases are studied previously.
(1) $a/c \to 0$. $c \to \infty$. Let

$$I_{1\mathrm{I}}^{**} = \lim_{(a/c)\to 0} I_1^{**}, \quad I_{2\mathrm{I}}^{**} = \lim_{(a/c)\to 0} I_2^{**}, \quad I_{\mathrm{I}}^{**} = \lim_{(a/c)\to 0} I^{**}, \quad (28)$$

$$B_{1\mathrm{I}} = \lim_{(a/c)\to 0} B_1, \quad B_{2\mathrm{I}} = \lim_{(a/c)\to 0} B_2, \quad B_{\mathrm{I}} = \lim_{(a/c)\to 0} B \quad (29)$$

$$w_{01\mathrm{I}} = \lim_{(a/c)\to 0} w_{01}, \quad \zeta_{\mathrm{I}} = \lim_{(a/c)\to 0} \zeta \quad (30)$$

From eqn (19) it can be seen that $\sin\psi = 1$ and $\cos\psi = 0$, so that from eqn (21), we have $I_i^* = 0$

In this extreme case, the crack can be considered as a through-width one, as shown in Fig. 5, then

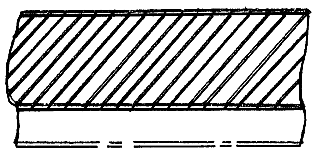

Fig. 5. Cracked section with $a/c \to 0$.

$$\zeta_{\mathrm{II}} = 0, \quad I^{**}_{\mathrm{I}} = I^{**}_{\mathrm{II}}, \quad B_{\mathrm{I}} = B_{\mathrm{II}} \tag{31}$$

So, eqn (27) becomes a differential equation of Bernouli's type. The solution is

$$\frac{1}{w_{01\mathrm{I}}} = e^{2g_1} B_{\mathrm{I}} \left\{ -\frac{\mu}{4\tau_0} \int_0^{g_1} \frac{1}{B_{\mathrm{I}}^2} \frac{I^{**}_{\mathrm{I}}}{a} e^{-2g^*_{\mathrm{I}}} \mathrm{d}g^*_{\mathrm{I}} + \Delta_{\mathrm{I}} \right\},$$

$$\Delta_{\mathrm{I}} = \left. \frac{1}{w_{01\mathrm{I}} B_{\mathrm{I}}} \right|_{g_1 = 0} \tag{32}$$

From the Appendix, it can also be obtained that

$$w_{01\mathrm{I}} = \frac{\tau_0}{2\mu} a F\left(\frac{a}{t}, 0\right) \tag{33}$$

(2) $c/a \to 0$. $a \to \infty$. Let

$$J^*_{1\mathrm{II}} = \lim_{(c/a)\to 0} J^*_1, \quad J^*_{2\mathrm{II}} = \lim_{(c/a)\to 0} J^*_2, \quad J^*_{\mathrm{II}} = \lim_{(c/a)\to 0} J^*, \tag{34}$$

$$B_{1\mathrm{II}} = \lim_{(c/a)\to 0} B_1, \quad B_{2\mathrm{II}} = \lim_{(c/a)\to 0} B_2, \quad B_{\mathrm{II}} = \lim_{(c/a)\to 0} B, \tag{35}$$

$$w_{01\mathrm{II}} = \lim_{(c/a)\to 0} w_{01}, \quad \zeta_{\mathrm{II}} = \lim_{(c/a)\to 0} \zeta, \tag{36}$$

From eqn (19), it can be seen that $\sin\psi = 0$ and $\cos\psi = 1$, so that from eqn (21), we have $J^{**}_i = 0$.

In this case, the crack can be considered as a through-thickness one which is unsymmetric with respect to the axis, as shown in Fig. 6, then

$$\zeta_{\mathrm{II}} = \zeta^*, \quad J^*_{\mathrm{II}} = J^*_{1\mathrm{II}} + \zeta^{*2} J^*_{2\mathrm{II}} \quad B_{\mathrm{II}} = B_{1\mathrm{II}} + \zeta^* B_{2\mathrm{II}} \tag{37}$$

where ζ^* is the ratio of w_{02} and w_{01} in the case of 2-D through the thickness crack, and can be found from the Appendix.

So, eqn (27) also becomes a differential equation of Bernouli's type. The solution is

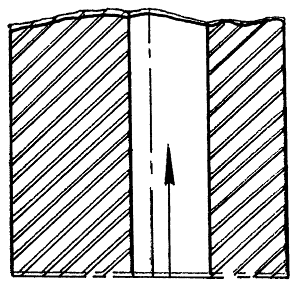

Fig. 6. Cracked section when $c/a \to 0$.

$$\frac{1}{w_{01\text{II}}} = e^{2g_1}B_{\text{II}}\left\{-\frac{E_1}{8\tau_0}\int_0^{g_1}\frac{1}{B_{\text{II}}^2}\frac{J_{\text{II}}^*}{c}e^{-2g^*}\,\mathrm{d}g^* + \Delta_2\right\}$$

$$\Delta_2 = \left.\frac{1}{w_{01\text{II}}B_{\text{II}}}\right|_{g_1=0} \tag{38}$$

From the Appendix, it can also be obtained that

$$w_{01\text{II}} = \frac{\tau_0}{E}cF\left(\frac{c}{b},\frac{f_z}{b}\right) \tag{39}$$

Let

$$I = I^* + \frac{2\mu}{E_1}I^{**}, \quad J = J^* + \frac{2\mu}{E_1}J^{**} \tag{40}$$

For the case of arbitrary a/c, an assumption of variable separation can be made as follows:

$$\frac{I + \frac{a}{c}J}{B^2} = m\left(\frac{a}{c},\frac{t}{b}\right)\frac{2\mu}{E_1}\frac{I_{\text{I}}^{**}}{B_{\text{I}}^2} + n\left(\frac{c}{a},\frac{b}{t}\right)\frac{a}{c}\frac{J_{\text{II}}^*}{B_{\text{II}}^2}$$

$$= \frac{2\mu}{E_1}m\left(\frac{a}{c},\frac{t}{b}\right)\frac{I_{\text{II}}^{**}}{B_{\text{II}}^2} + n\left(\frac{a}{c},\frac{b}{t}\right)\frac{a}{c}(J_{\text{II}}^* + \zeta^{*2}J_{2\text{II}}^*)/(B_{1\text{II}}^* + \zeta^* B_{2\text{II}}^*) \tag{41}$$

where $m(a/c, t/b)$ and $n(c/a, b/t)$ can be determined by least square method in symmetric cases. The above assumption can be verified numerically.

After substitution, eqn (27) will be transformed into the following

$$\frac{\partial w_{01}}{\partial g_1} + \left(2 + \frac{1}{B}\frac{\partial B}{\partial g_1}\right)w_{01} = \frac{B}{8\tau_0}\left[m\left(\frac{a}{c},\frac{t}{b}\right)\frac{2\mu}{a}\frac{I_{\text{I}}^{**}}{B_{\text{I}}^2} + n\left(\frac{c}{a},\frac{b}{t}\right)\frac{E_{\text{I}}}{c}\frac{J_{\text{II}}^*}{B_{\text{II}}^2}\right]w_{01}^2 \tag{42}$$

The above equation is a special case of classical Bernouli's equation. The solution of eqn (42) is

$$\frac{1}{w_{01}} = e^{2g_1}B\left\{-\frac{1}{8\tau_0}\int_0^{g_1}\left[m\left(\frac{a}{c},\frac{t}{b}\right)\frac{2\mu I_{\text{I}}^{**}}{aB_{\text{I}}^2} + n\left(\frac{a}{c},\frac{b}{t}\right)\frac{E_1}{c}\frac{J_{\text{II}}^*}{B_{\text{II}}^2}\right]e^{-2g_1^*}\,\mathrm{d}g_1^* + \Delta\right\} \tag{43}$$

where, $\Delta = (1/w_{01}B)|_{g=0}$.

Comparing eqn (43) with (32) and (38), and considering the state function property of w_{01}, we have

$$\frac{1}{w_{01}} = m\left(\frac{a}{c},\frac{t}{b}\right)\frac{B}{w_{01\text{I}}B_{\text{I}}} + n\left(\frac{c}{a},\frac{b}{t}\right)\frac{B}{w_{02\text{II}}B_{\text{II}}}$$

$$\Delta = m\Delta_1\frac{B}{B_{\text{I}}} + n\Delta_2\frac{B}{B_{\text{II}}} \tag{44}$$

So, the closed form solution of w_{01} is obtained as shown in eqn (44).

From (24), ζ will be equal to

$$\zeta = \frac{-\dfrac{\partial}{\partial g_2}\left(\dfrac{1}{w_{01}B}\right)}{\dfrac{E_1}{4\tau_0 B^2}\left(\dfrac{M^{*}+\dfrac{2\mu}{E_1}M^{**}}{a}+\dfrac{N^{*}+\dfrac{2\mu}{E_1}N^{**}}{c}\right)\dfrac{f_z}{c}} \tag{45}$$

To simplify the expression of ζ, the above two extreme cases should be considered once more:

(1) $a/c = 0$.

From eqn (45), we have,

$$\zeta_{\mathrm{I}} = \frac{-\dfrac{\partial}{\partial g_2}\left(\dfrac{1}{w_{01\mathrm{I}}B_{\mathrm{I}}}\right)}{\dfrac{E_1}{4\tau_0 B_{\mathrm{I}}^2}\cdot\dfrac{M_{\mathrm{I}}^{*}+\dfrac{2\mu}{E_1}M_{\mathrm{I}}^{**}}{a}\cdot\dfrac{f_z}{c}} \tag{46}$$

(2) $c/a \to 0$.

From eqn (45), we have

$$\zeta_{\mathrm{II}} = \frac{-\dfrac{\partial}{\partial g_2}\left(\dfrac{1}{w_{01\mathrm{II}}B_{\mathrm{II}}}\right)}{\dfrac{E_1}{4\tau_0 B_{\mathrm{II}}^2}\cdot\dfrac{N_{\mathrm{II}}^{*}+\dfrac{2\mu}{E_1}N_{\mathrm{II}}^{**}}{c}\cdot\dfrac{f_z}{c}} \tag{47}$$

Substituting eqn (44) into eqn (45), and considering eqns (46) and (47), it can be obtained that

$$\zeta = \frac{m\left(\dfrac{a}{c},\dfrac{t}{b}\right)\dfrac{M_{\mathrm{I}}}{a}\zeta_{\mathrm{I}}+n\left(\dfrac{c}{a},\dfrac{b}{t}\right)\dfrac{N_{\mathrm{II}}}{c}\zeta_{\mathrm{II}}}{\dfrac{M}{a}+\dfrac{N}{c}} \tag{48}$$

where

$$M = M^{*}+\frac{2\mu}{E_1}M^{**},\quad N = N^{*}+\frac{2\mu}{E_1}N^{**}$$

$$M_{\mathrm{I}} = M_{\mathrm{I}}^{*}+\frac{2\mu}{E_1}M_{\mathrm{I}}^{**},\quad N_{\mathrm{II}} = N_{\mathrm{II}}^{*}+\frac{2\mu}{E_1}N_{\mathrm{II}}^{**} \tag{49}$$

Substituting eqns (31) and (37) into (48), we have

$$\zeta = n\left(\frac{a}{c},\frac{b}{t}\right)\cdot\frac{\frac{a}{c}N_{\mathrm{II}}}{M+\frac{a}{c}N}\zeta_2^* \tag{50}$$

Finally, the closed form solution of w_{0i} are obtained, as shown in eqns (44) and (50). If the modes of the crack surface displacement are known, K_{II} and K_{III} can be calculated from eqns (11) and (12).

5. Modes of crack surface displacements

It can be assumed that, the symmetric and anti-symmetric modes of crack surface displacement of each section about 3-D cracked body are the same as those of a corresponding 2-D cracked plate with the same geometrical configuration and the same type of load distribution along the crack surface, respectively, given in the Appendix.

Let $A(x_0,z_0)$ be an arbitrary point in the crack area as shown in Fig. 7. If $w_1(z_0,x)$ and $w_2(x_0,z)$ are 2-D crack surface displacements (shown in the Appendix) used to express the crack surface displacements $w(x,z_0)$ and $w(x_0,z)$ of transversal and longitudinal sections about a 3-D cracked body, respectively, then from (A43) and (A63) we have

$$\begin{aligned} w(x,z_0) &= w_1(z_0,x) \\ &= w_{011}(z_0)\left\{n_1\alpha_1\left(\frac{a'}{t}\right)\left[\beta_1\left(\frac{x}{a'}\right)\right]^{1/2}+\left[1-n_1\alpha_1\left(\frac{a'}{t}\right)\right]\left[\beta_1\left(\frac{x}{a'}\right)\right]^{3/2}\right\} \end{aligned} \tag{51}$$

$$\begin{aligned} w(x_0,z) &= w_2(x_0,z) \\ &= w_{021}(x_0)\left\{n_2\alpha_2\left(\frac{c_1'}{b},\frac{c_2'}{b}\right)\left[\beta_2\left(\frac{z}{c'}\right)\right]^{1/2}+\left[1-n_2\alpha_2\left(\frac{c_1'}{b},\frac{c_2'}{b}\right)\right]\left[\beta_2\left(\frac{z}{c'}\right)\right]^{3/2}\right\} \\ &\quad +w_{022}(x_0)\left[\beta_2\left(\frac{z}{c'}\right)\right]^{1/2}\frac{z}{c'} \end{aligned} \tag{52}$$

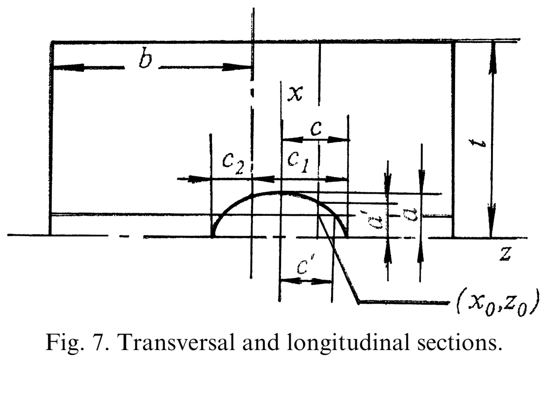

Fig. 7. Transversal and longitudinal sections.

where, w_{011} is the generalized crack surface displacements of transversal section through point $A(x_0, z_0)$. w_{021} and w_{022} are the generalized crack surface displacements of longitudinal section through point $A(x_0, z_0)$ also for the symmetric and antisymmetric case, respectively. n_i and β_i can be found from the Appendix, and we have

$$n_1 = 2, \quad \beta_1\left(\frac{x}{a}\right) = 1 - \frac{x^2}{a^2}, \quad \text{for embedded crack}$$

$$n_1 = 4, \quad \beta_1\left(\frac{x}{a}\right) = 1 - \frac{x}{a}, \quad \text{for surface crack}$$

$$n_2 = 2, \quad \beta_2\left(\frac{z}{c}\right) = 1 - \frac{z^2}{c^2} \tag{53}$$

To satisfy the requirements about compatibility between crack surface displacements of transversal sections and those of longitudinal ones along the x-axis and z-axis, the following equalities must be valid

$$\begin{aligned} w_1(z,0) &= w_2(0,z) \\ w_1(0,x) &= w_2(x,0) \\ \frac{d}{dz} w_1(z,x)\bigg|_{z=0} &= \frac{d}{dz} w_2(x,z)\bigg|_{z=0} \end{aligned} \tag{54}$$

Substituting eqn (51) and eqn (52) into eqn (54), it can be obtained that

$$\begin{aligned} w_{011}(z) = w_{01} & \left\{ n_2\alpha_2\left(\frac{c_1}{b}, \frac{c_2}{b}\right)\left[\beta_2\left(\frac{z}{c}\right)\right]^{1/2} + \left[1 - n_2\alpha_2\left(\frac{c_1}{b}, \frac{c_2}{b}\right)\right]\left[\beta_2\left(\frac{z}{c}\right)\right]^{3/2} \right\} \\ & + w_{02}\left[\beta_2\left(\frac{z}{c}\right)\right]^{1/2}\frac{z}{c} \end{aligned}$$

$$w_{021}(x) = w_{01}\left\{ n_1\alpha_1\left(\frac{a}{t}\right)\left[\beta_1\left(\frac{x}{a}\right)\right]^{1/2} + \left[1 - n_1\alpha_1\left(\frac{a}{t}\right)\right]\left[\beta_1\left(\frac{x}{a}\right)\right]^{3/2} \right\} \tag{55}$$

$$w_{022}(x) = w_{02}\left\{ n_1\alpha_1\left(\frac{a}{t}\right)\left[\beta_1\left(\frac{x}{a}\right)\right]^{1/2} + \left[1 - n_1\alpha_1\left(\frac{a}{t}\right)\right]\left[\beta_1\left(\frac{x}{a}\right)\right]^{3/2} \right\} \tag{56}$$

Then, we have

$$\begin{aligned} w_1(z,x) &= w_{01}h_{11}(z,x) + w_{02}h_{12}(x,z) \\ w_2(x,z) &= w_{01}h_{21}(x,z) + w_{02}h_{22}(x,z) \end{aligned} \tag{57}$$

where,

$$h_{11}(x,z) = \left\{n_1\alpha_1\left(\frac{a'}{t}\right)\left[\beta_1\left(\frac{x}{a'}\right)\right]^{1/2} + \left[1-n_1\alpha_1\left(\frac{a'}{t}\right)\right]\left[\beta_1\left(\frac{x}{a'}\right)\right]^{3/2}\right\}$$
$$\times\left\{n_2\alpha_2\left(\frac{c_1}{b},\frac{c_2}{b}\right)\left[\beta_2\left(\frac{z}{c}\right)\right]^{1/2} + \left[1-n_2\alpha_2\left(\frac{c_1}{b},\frac{c_2}{b}\right)\right]\left[\beta_2\left(\frac{z}{c}\right)\right]^{3/2}\right\}$$
$$h_{12}(x,z) = \left\{n_1\alpha_1\left(\frac{a'}{t}\right)\left[\beta_1\left(\frac{x}{a'}\right)\right]^{1/2} + \left[1-n_1\alpha_1\left(\frac{a'}{t}\right)\right]\left[\beta_1\left(\frac{x}{a'}\right)\right]^{3/2}\right\}\left[\beta_2\left(\frac{z}{c}\right)\right]^{1/2}\frac{z}{c}$$
$$h_{21}(x,z) = \left\{n_1\alpha_1\left(\frac{a}{t}\right)\left[\beta_1\left(\frac{x}{a}\right)\right]^{1/2} + \left[1-n_1\alpha_1\left(\frac{a}{t}\right)\right]\left[\beta_1\left(\frac{x}{a}\right)\right]^{3/2}\right\}$$
$$\times\left\{n_2\alpha_2\left(\frac{c_1'}{b},\frac{c_2'}{b}\right)\left[\beta_2\left(\frac{z}{c'}\right)\right]^{1/2} + \left[1-n_2\alpha_2\left(\frac{c_1'}{b},\frac{c_2'}{b}\right)\right]\left[\beta_2\left(\frac{z}{c'}\right)\right]^{3/2}\right\}$$
$$h_{22}(x,z) = \left\{n_1\alpha_1\left(\frac{a}{t}\right)\left[\beta_1\left(\frac{x}{a}\right)\right]^{1/2} + \left[1-n_1\alpha_1\left(\frac{a}{t}\right)\right]\left[\beta_1\left(\frac{x}{a}\right)\right]^{3/2}\right\}\left[\beta_2\left(\frac{z}{c'}\right)\right]^{1/2}\frac{z}{c'} \quad (58)$$

$$H_{ij}(x,z_1) = h_{1i}(x,z_1)h_{1j}(x,z_1)$$
$$H_{ij}(x_1,z) = h_{2i}(x_1,z)h_{2j}(x_1,z) \quad (59)$$
$$h_i(x,z) = [h_{1i}(x,z)+h_{2j}(x,z)/2 \quad (60)$$

Substituting H_{ij} and h_1 into the expressions of w_{0i}, K_{II} and K_{III}, the stress intensity factors K_{II} and K_{III} can be obtained.

6. Results

For 3-D finite bodies with eccentric cracks subjected to shear loading, as shown in Fig. 8, a series of new results are obtained. Some of the results are given in Figs 9–10. When $f_z = 0$, the case

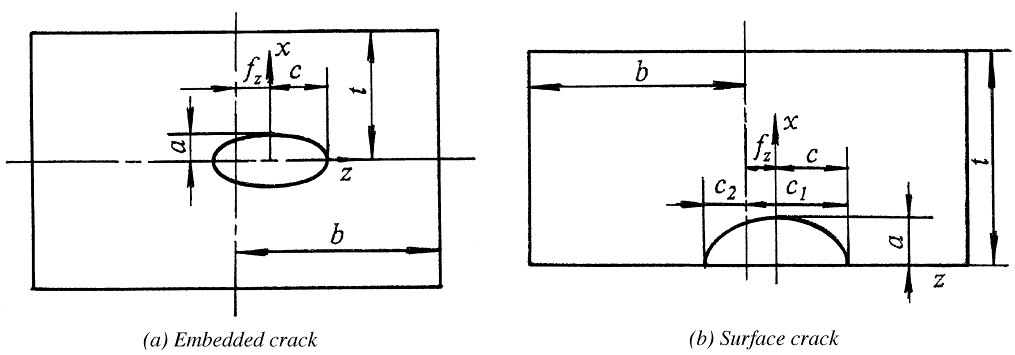

Fig. 8. Two typical cases. (a) Embedded crack. (b) Surface crack.

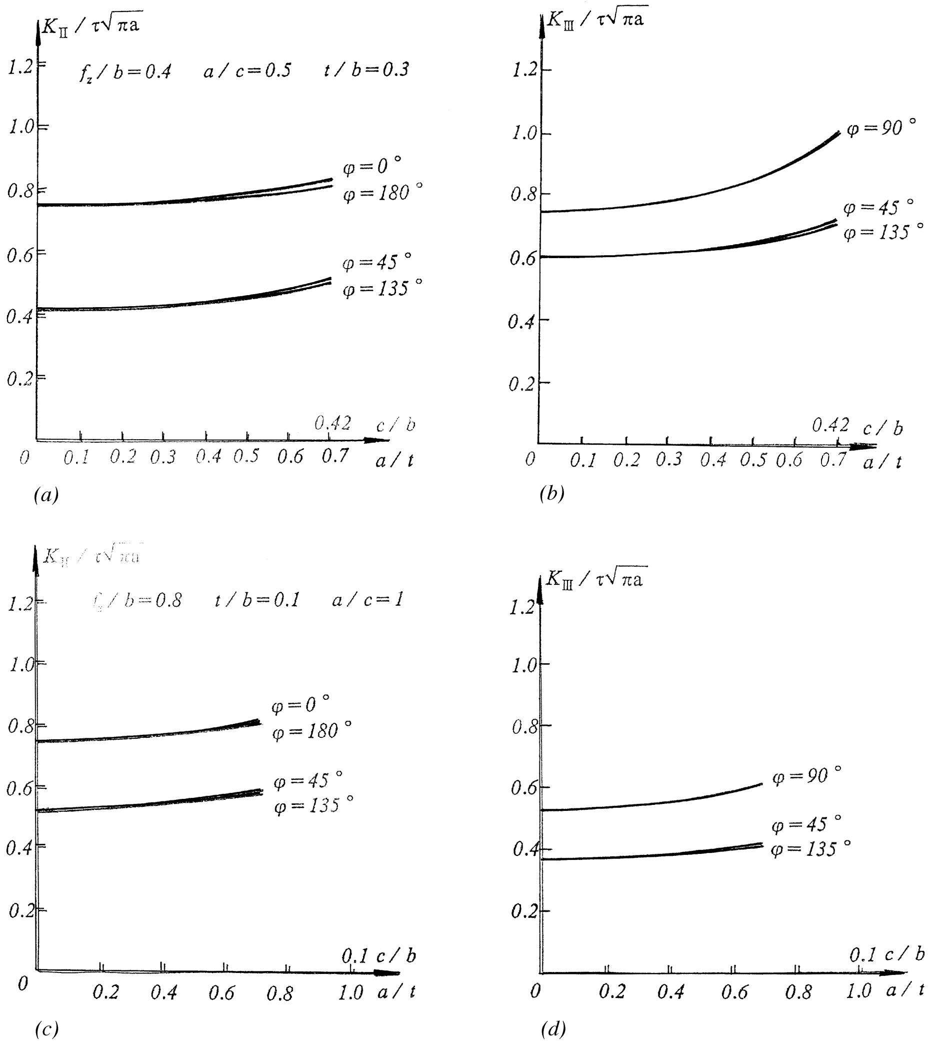

Fig. 9. Stress intensity factors of shear modes for embedded eccentric crack.

degenerates into a symmetric one, and the results fit quite well with the existing ones, as shown in Fig. 11.

7. Conclusions

From the above derivations and computations, the following conclusions can be obtained

(1) A Pythagorean theorem to show the relationship among the three-dimensional crack surface displacements and the crack sliding displacements of longitudinal section and the crack tearing displacements of transversal sections in the vicinity of the crack front can be established.

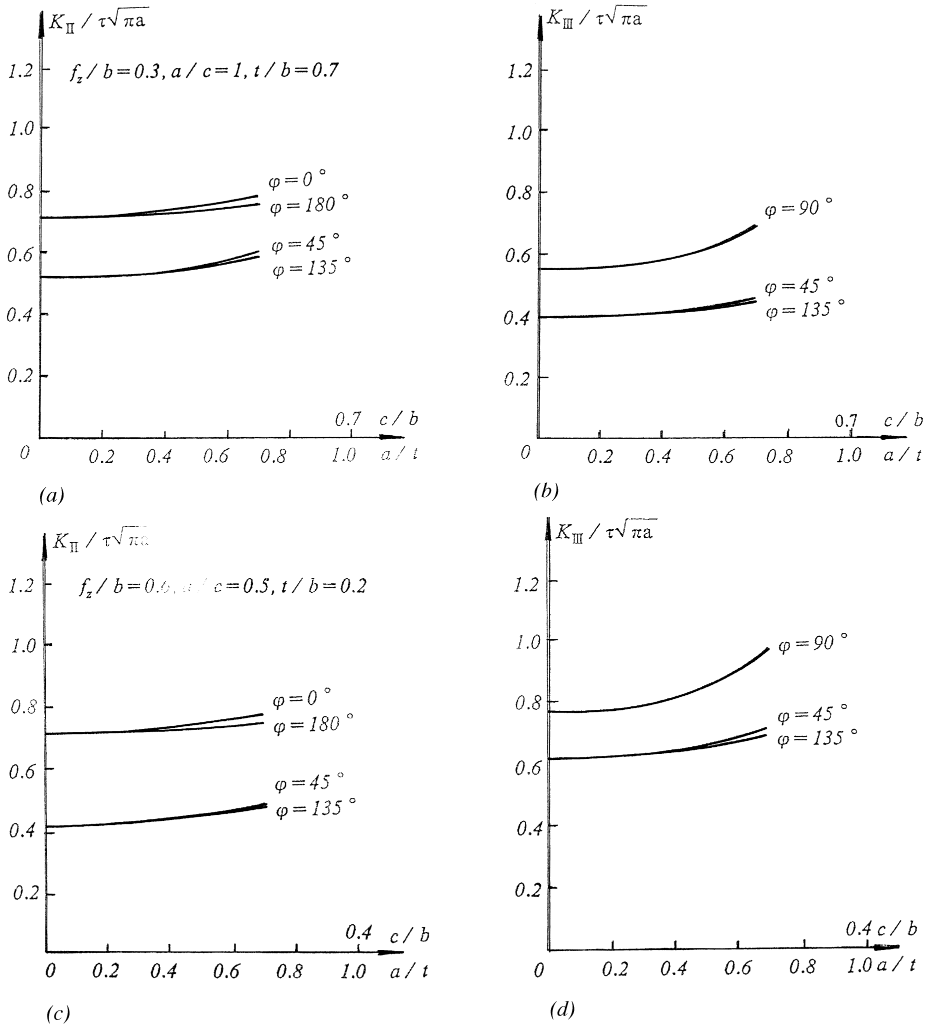

Fig. 10. Stress intensity factors of shear modes for surface eccentric crack.

(2) The energy release rate method can be used to formulate a system of classical Bernouli's equation about the generalized 3-D crack surface displacements. The above equations can be solved in closed form using the assumption of separation of variables.
(3) Three assumptions used in this paper (generalized Young's modulus, mode of crack surface displacement and separation of variables) can be verified numerically.
(4) The calculation is very time-saving, as the main work is only to calculate several integrals numerically. For any given case, the stress intensity factors along the crack front can be calculated within three seconds of C.P.U. time on IBM 4341. Hence a complete series of useful results about the stress intensity factors can be obtained.
(5) The results provided by this method are in nice agreement with those obtained by other methods, in symmetric cases.

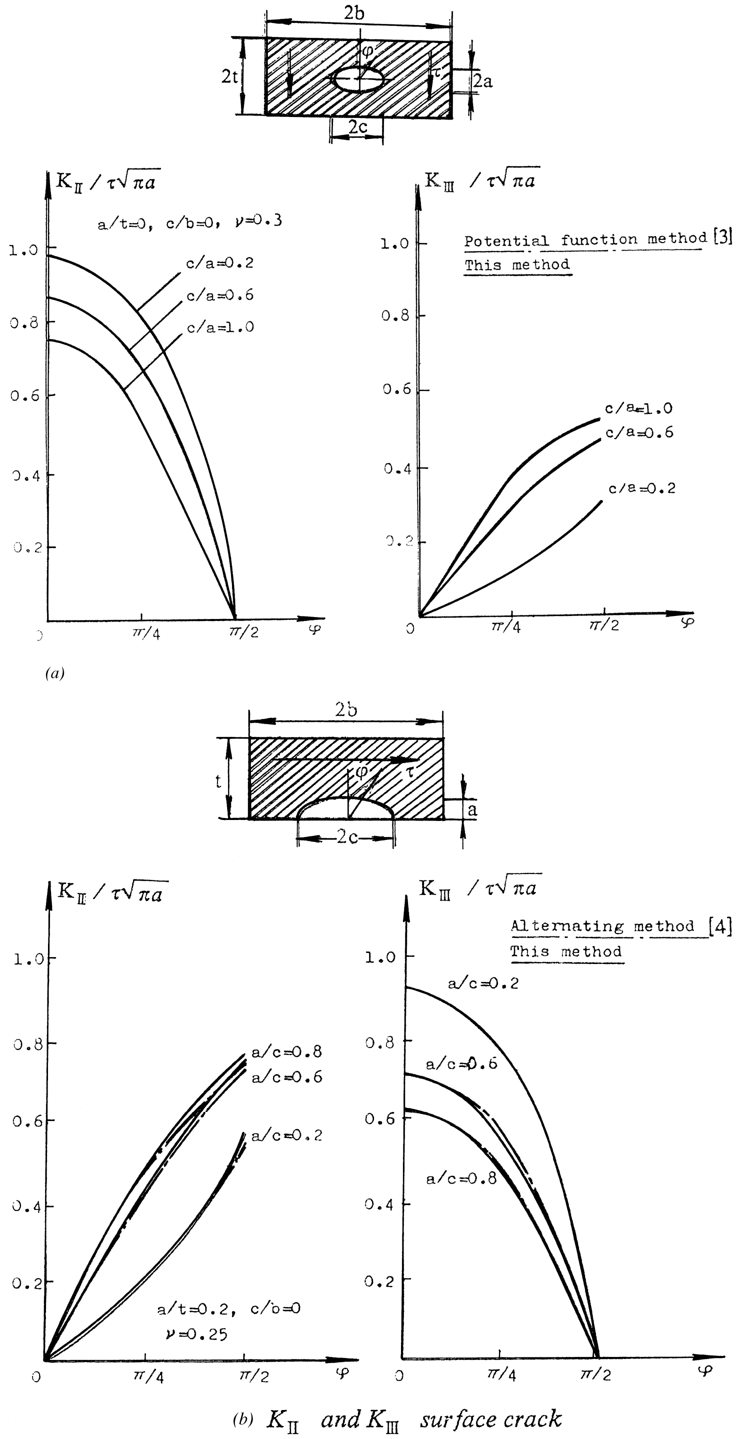

Fig. 11. Comparison about results of this method and those of other methods. (a) K_{II} and K_{III} of embedded crack (completely fit with the exact results of potential function method for infinite bodies). (b) K_{II} and K_{III} surface crack.

(6) In general, the 3-D stress intensity factors can be obtained by energy release rate method, if the stress intensity factors of the longitudinal and transversal sections in the 2-D case are known.

Appendix. The crack surface displacement of 2-D cracks

A1. The in-plane sliding displacement of 2-D unsymmetric inner crack

*A*1.1. *Analytical expressions*

For plane problems in theory of elasticity, the stress and displacement components can be obtained as follows:

$$\sigma_{yy} - i\sigma_{xy} = \varphi'(z) + \overline{\Omega'(z)} + (z - \bar{z})\overline{\varphi''(z)} \tag{A1}$$

$$2\mu(u_x + iu_y) = \kappa\varphi(z) - \overline{\Omega(z)} - (z - \bar{z})\bar{\varphi}'(z) \tag{A2}$$

For an inner crack as shown in Fig. 1A, as the crack surface is traction-free, we have

$$\varphi'(z) = (z^2 - a^2)^{-(1/2)} \sum_{m=0}^{M} F_m z^m + \sum_{n=0}^{N} G_n z^n$$

$$\Omega'(z) = (z^2 - a^2)^{-(1/2)} \sum_{m=0}^{M} \bar{F}_m z^m - \sum_{n=0}^{N} \bar{G}_n z^n \tag{A3}$$

Let

$$\beta_m = \int (z^2 - a^2)^{-(1/2)} z^m \, dz \tag{A4}$$

Then, it can be obtained that

$$\beta_0 = \ln(z + \sqrt{z^2 - a^2}), \quad \beta_1 = \sqrt{z^2 - a^2},$$

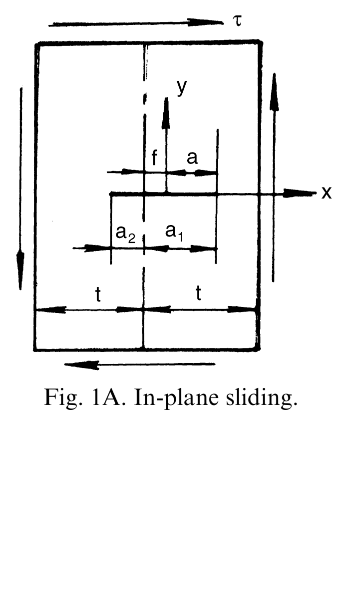

Fig. 1A. In-plane sliding.

$$\beta_n = \frac{z^{n-1}}{n}\sqrt{z^2-a^2}+\frac{n-1}{n}\alpha^2\beta_{n-2} \tag{A5}$$

So, β_n can be written as follows:

$$\beta_{2t-1} = \sum_{j=1}^{t} b_{2j-1,2t-1}(z^2-a^2)^{(2j-1)/2}$$

$$\beta_{2t} = \sum_{j=1}^{t} b_{2j,2t}z(z^2-a^2)^{(2j-1)/2}+b_{0,2t}\ln(z+\sqrt{z^2-a^2}) \tag{A6}$$

where, b_{ij} is a known constant obtained from the recurrence formula (A5).

From (A3), $\varphi(z)$ and $\Omega(z)$ can be expressed as follows:

$$\varphi(z) = \sum_{m=0}^{M} F_m\beta_m(z)+\sum_{n=0}^{N} G_n\frac{z^{n+1}}{n+1}$$

$$\Omega(z) = \sum_{m=0}^{M} \bar{F}_m\beta_m(z)-\sum_{n=0}^{N} \bar{G}_n\frac{z^{n+1}}{n+1} \tag{A7}$$

in which

$$\sum_{m=0}^{M} F_m\beta_m(z) = f_0\ln(z+\sqrt{z^2-a^2})+\sum_{t=1}^{[M/2]} f_{2t-1}(z^2-a^2)^{(2t-1)/2}+\sum_{t=1}^{[M/2]} f_{2t}(z^2-a^2)^{(2t-1)/2} \tag{A8}$$

and

$$f_0 = \sum_{f=0}^{[M/2]} F_{2f}b_{0,2f}, \quad f_{2f} = \sum_{f=0}^{[M/2]} F_{2f}b_{2t,2f}, \quad f_{2f-1} = \sum_{j=1}^{[M/2]} F_{2f-1}b_{2t-1,2f-1} \tag{A9}$$

When M is even, $[M/2] = M/2$. When M is odd, $[M/2] = (M+1)/2$, for the summation of odd terms: $[M/2] = (M-1)/2$, for the summation of even terms. Substituting (A7) and (A8) into (A2), the complex displacement expression can be obtained in a series form

$$\begin{aligned} 2\mu(u_x+iu_y) &= f_0\{\kappa\ln(z+\sqrt{z^2-a^2}-\ln(\bar{z}+\sqrt{\bar{z}^2-a^2})\} \\ &+\sum_{t=1}^{[M/2]} f_{2t-1}\{\kappa(z^2-a^2)^{(2t-1)/2}-(\bar{z}^2-a^2)^{(2t-1)/2}\} \\ &+\sum_{t=1}^{[M/2]} f_{2t}\{\kappa z(z^2-a^2)^{(2t-1)/2}-\bar{z}(\bar{z}^2-a^2)^{(2t-1)/2}\} \\ &-(z-\bar{z})\left\{\sum_{m=0}^{M}\bar{F}_m(\bar{z}^2-a^2)^{1/2}\bar{z}^m+\sum_{n=}^{N}\bar{G}_n\bar{z}^n\right\}+\sum_{m=0}^{N} G_n\left\{\kappa\frac{z^{n+1}}{n+1}-\frac{\bar{z}^{n+1}}{n+1}\right\} \end{aligned} \tag{A10}$$

The first term of the above equation is a multi-valued function. By means of single-valued condition of displacement, $f_0 = 0$. Furthermore, when z changes from the upper surface of the crack to its lower surface, the crack opening and sliding displacements have the character of antisymmetric discontinuity. So, there will be no G_n terms. Then, the crack surface displacement can be expressed as follows:

$$(u_x+iu_y)|_{\text{crack surface}} = \frac{\kappa+1}{2\mu}\left\{\sum_{t=1}^{[M/2]} f_{2t-1}a^{2t-1}\left(1-\frac{x^2}{a^2}\right)^{(2t-1)/2} + \sum_{t=1}^{[M/2]} f_{2t}a^{2t}\frac{x}{a}\left(1-\frac{x^2}{a^2}\right)^{(2t-1)/2}\right\} \quad \text{(A11)}$$

where the first and second summation are the symmetric and antisymmetric parts of crack surface displacement with respect to y-axis, respectively.

According to the principle of superposition, it can be proved that the above expression (A11) can be used to describe the crack surface displacement of the same plate subjected to surface tractions along the crack surfaces which is corresponding to the load acting at the remote sides.

For an infinite plate subjected to uniform load, it is well known that, along the surface of the crack,

$$u_x = \frac{\kappa+1}{2\mu}\tau\cdot\frac{a}{2}\sqrt{1-\frac{x^2}{a^2}} = \frac{\tau}{E}a\sqrt{1-\frac{x^2}{a^2}} \quad \text{(A12)}$$

$$u_y = \frac{\kappa+1}{2\mu}\sigma\cdot\frac{a}{2}\sqrt{1-\frac{x^2}{a^2}} = \frac{\sigma}{E}a\sqrt{1-\frac{x^2}{a^2}} \quad \text{(A13)}$$

For finite plates, more terms are needed. In this paper, we choose three terms.

The determination of the crack opening displacement u_y was discussed in the Appendix of Wang et al. (1990a). Similarly, we will discuss the crack sliding displacement $w = u_x$.

Along the crack surface, the non-uniform shear load is assumed to be

$$\tau = \tau_0 s(z) \quad \text{(A14)}$$

The crack sliding displacement of the finite plate can be expressed as follows:

$$w = \frac{\tau_0}{E}a\left\{2f\left(\frac{a_1}{t},\frac{a_2}{t}\right)\left(1-\frac{x^2}{a^2}\right)^{1/2} + 2g\left(\frac{a_1}{t},\frac{a_2}{t}\right)\left(1-\frac{x^2}{a^2}\right)^{3/2}\right\} + \frac{\tau_0}{E}a2h\left(\frac{a_1}{t},\frac{a_2}{t}\right)\left(1-\frac{x^2}{a^2}\right)^{1/2}\frac{x}{a} \quad \text{(A15)}$$

On the right-hand side of the above equation, the first kind of terms and second kind of terms are used to denote the symmetric and antisymmetric displacements, respectively. f, g and h are unknown functions.

From (A15), the crack sliding displacement at origin w_{01} is equal to

$$w_{01} = \frac{\tau_0}{E}a\left[2f\left(\frac{a_1}{t},\frac{a_2}{t}\right)+2g\left(\frac{a_1}{t},\frac{a_2}{t}\right)\right] = \frac{\tau_0}{E}aF\left(\frac{a_1}{t},\frac{a_2}{t}\right) \quad \text{(A16)}$$

where

$$F\left(\frac{a_1}{t},\frac{a_2}{t}\right) = 2f\left(\frac{a_1}{t},\frac{a_2}{t}\right)+2g\left(\frac{a_1}{t},\frac{a_2}{t}\right) \tag{A17}$$

Furthermore, the crack sliding displacement gradient at origin will be

$$w_{02} = \left.\frac{\mathrm{d}w}{\mathrm{d}\left(\dfrac{x}{a}\right)}\right|_{(x/a)=0} = \frac{\tau_0}{E}a2h\left(\frac{a_1}{t},\frac{a_2}{t}\right) \tag{A18}$$

Let

$$\alpha\left(\frac{a_1}{t},\frac{a_2}{t}\right) = f\left(\frac{a_1}{t},\frac{a_2}{t}\right)\Big/ F\left(\frac{a_1}{t},\frac{a_2}{t}\right) \tag{A19}$$

Then substituting (A19) into (A15), we have

$$w = w_{01}\left\{2\alpha\left(\frac{a_1}{t},\frac{a_2}{t}\right)\left(1-\frac{x^2}{a^2}\right)^{1/2}+\left[1-2\alpha\left(\frac{a_1}{t},\frac{a_2}{t}\right)\right]\left(1-\frac{x^2}{a^2}\right)^{3/2}\right\} + w_{02}\left(1-\frac{x^2}{a^2}\right)^{1/2}\frac{x}{a} \tag{A20}$$

From (A20), the crack sliding displacement is determined by w_{01}, w_{02} and $a(a_1/t, a_2/t)$, and furthermore they are determined by $f(a_1/t, a_2/t)$, $F(a_1/t, a_2/t)$ and $h(a_1/t, a_2/t)$.

*A*1.2. *Determination of* f(a_1/t, a_2/t), F(a_1/t, a_2/t) *and* h(a_1/t, a_2/t)

Now, we are going to determine $f(a_1/t, a_2/t)$, $F(a_1/t, a_2/t)$ and $h(a_1/t, a_2/t)$. In the vicinity of right crack-tip, $x_R/a = 1-r/a$. In the vicinity of the left crack-tip, $x_L/a = -(1-r/a)$. Then, from (A15), it can be obtained that

$$w_R = \frac{\tau_0}{E}a2\sqrt{2r}\left[f\left(\frac{a_1}{t},\frac{a_2}{t}\right)+h\left(\frac{a_1}{t},\frac{a_2}{t}\right)\right]$$

$$w_L = \frac{\tau_0}{E}a2\sqrt{2r}\left[f\left(\frac{a_1}{t},\frac{a_2}{t}\right)-h\left(\frac{a_1}{t},\frac{a_2}{t}\right)\right] \tag{A21}$$

On the other hand, in the vicinity of crack-tip, it is well known that

$$w(r) = \sqrt{\frac{8}{\pi}}\frac{K_{\mathrm{II}}}{E}\sqrt{r} \tag{A22}$$

It is assumed that the stress intensity factors are given and can be written as follows:

$$K_{\mathrm{II}R} = \eta_R\left(\frac{a_1}{t},\frac{a_2}{t}\right)\tau_0\sqrt{\pi a}$$

$$K_{\mathrm{II}L} = \eta_L\left(\frac{a_1}{t},\frac{a_2}{t}\right)\tau_0\sqrt{\pi a} \tag{A23}$$

In the above expression, factors $\eta_R(a_1/t, a_2/t)$, $\eta_L(a_1/t, a_2/t)$ are given functions of a_1/t and a_2/t.

Substituting (A23) into (A22), and comparing the results with (A21), it can be obtained that

$$f\left(\frac{a_1}{t},\frac{a_2}{t}\right) = \left[\eta_R\left(\frac{a_1}{t},\frac{a_2}{t}\right)+\eta_L\left(\frac{a_1}{t},\frac{a_2}{t}\right)\right]\Big/ 2 \tag{A24}$$

$$h\left(\frac{a_1}{t},\frac{a_2}{t}\right) = \left[\eta_g\left(\frac{a_1}{t},\frac{a_2}{t}\right)-\eta_L\left(\frac{a_1}{t},\frac{a_2}{t}\right)\right]\Big/ 2 \tag{A25}$$

So, we have

$$\alpha\left(\frac{a_1}{t},\frac{a_2}{t}\right) = \left[\eta_R\left(\frac{a_1}{t},\frac{a_2}{t}\right)+\eta_L\left(\frac{a_1}{t},\frac{a_2}{t}\right)\right]/2F\left(\frac{a_1}{t},\frac{a_2}{t}\right) \tag{A26}$$

$$w_{01} = \frac{\tau_0}{E}aF\left(\frac{a_1}{t},\frac{a_2}{t}\right) \tag{A27}$$

$$w_{02} = w_{01}\left[\eta_R\left(\frac{a_1}{t},\frac{a_2}{t}\right)-\eta_L\left(\frac{a_1}{t},\frac{a_2}{t}\right)\right]\Big/ F\left(\frac{a_1}{t},\frac{a_2}{t}\right) \tag{A28}$$

$$\zeta = \frac{w_{02}}{w_{01}} = \left[\eta_R\left(\frac{a_1}{t},\frac{a_2}{t}\right)-\eta_L\left(\frac{a_1}{t},\frac{a_2}{t}\right)\right]\Big/ F\left(\frac{a_1}{t},\frac{a_2}{t}\right) \tag{A29}$$

When there is a crack propagation da, then from (A16) the increment of w_{01} will be

$$\mathrm{d}w_{01} = L\frac{w_{01}}{a}\mathrm{d}a \tag{A30}$$

where

$$L = 1+\left[\mathrm{d}F\left(\frac{a_1}{t},\frac{a_2}{t}\right)\Big/ \mathrm{d}a\right]\Big/ F\left(\frac{a_1}{t},\frac{a_2}{t}\right) \tag{A31}$$

Now, it is only necessary to determine $F(a_1/t, a_2/t)$ by energy release rate method. For a plate of unit thickness with an inner crack, we assume that the virtual crack extension is a proportional one. Then,

$$\mathrm{d}a = \mathrm{d}a_1 = \mathrm{d}a_2 \tag{A32}$$

The total potential energy release rate is

$$\frac{\mathrm{d}\Pi}{\mathrm{d}a} = -\left(\frac{K_{\mathrm{II}R}^2}{E}+\frac{K_{\mathrm{II}L}^2}{E}\right) = -\frac{\pi\tau_0^2 a}{E}\left[\eta_R^2\left(\frac{a_1}{t},\frac{a_2}{t}\right)+\eta_L^2\left(\frac{a_1}{t},\frac{a_2}{t}\right)\right] \tag{A33}$$

Then,

$$\Pi = -\frac{\pi\tau_0^2 a^2}{E\left(\frac{a}{t}\right)^2}\left\{\int_0^{a/t}\frac{a}{t}\left[\eta_R^2\left(\frac{a_1}{t},\frac{a_2}{t}\right)+\eta_L^2\left(\frac{a_1}{t},\frac{a_2}{t}\right)\right]\mathrm{d}\left(\frac{a}{t}\right)+\Delta\right\} \tag{A34}$$

On the other hand, according to the definition of total potential energy Π, we have

$$\Pi = -\tau_0\int_{-a}^{a} w(x)s(x)\,\mathrm{d}x \tag{A35}$$

Inserting (A15) into (A35) it can be obtained that

$$\Pi = \frac{2\tau_0^2 a^2}{E}\left\{\left[2f\left(\frac{a_1}{t},\frac{a_2}{t}\right)(m_1-m_3)+F\left(\frac{a_1}{t},\frac{a_2}{t}\right)\cdot m_3\right]+2h\left(\frac{a_1}{t},\frac{a_2}{t}\right)\cdot n_1\right\} \tag{A36}$$

where,

$$m_1 = \frac{1}{2}\int_{-1}^{1}\left(1-\frac{x^2}{a^2}\right)^{1/2} s(x)\,\mathrm{d}\frac{x}{a},\quad m_3 = \frac{1}{2}\int_{-1}^{1}\left(1-\frac{x^2}{a^2}\right)^{3/2} s(x)\,\mathrm{d}\frac{x}{a}$$

$$n_1 = \frac{1}{2}\int_{-1}^{1}\left(1-\frac{x^2}{a^2}\right)^{1/2}\frac{x}{a}s(x)\,\mathrm{d}\frac{x}{a} \tag{A37}$$

Comparing (A34) and (A36), it can be known that

$$F\left(\frac{a_1}{t},\frac{a_2}{t}\right) = \frac{\pi}{2m_3\cdot\left(\frac{a}{t}\right)^2}\left\{\int_0^{a/t}\frac{a^*}{t}\left[\eta_R^2\left(\frac{a_1^*}{t},\frac{a_2^*}{t}\right)+\eta_L^2\left(\frac{a_1^*}{t},\frac{a_2^*}{t}\right)\right]\mathrm{d}\frac{a^*}{t}+\Delta\right\}$$
$$-\left(\frac{m_1}{m_3}-1\right)\cdot 2f\left(\frac{a_1}{t},\frac{a_2}{t}\right)-\frac{n_1}{m_3}\cdot 2h\left(\frac{a_1}{t},\frac{a_2}{t}\right) \tag{A38}$$

To determine the constant Δ, let us consider the cracked plate of infinite width as the initial condition. Under this condition, from eqn (A16), it can be known that $F(a_1/t, a_2/t)$ is a finite parameter when $a/c \to 0$. So that

$$\lim_{(a/t)\to 0}\left(\frac{a}{t}\right)^2 F\left(\frac{a_1}{t},\frac{a_2}{t}\right) = 0 \tag{A39}$$

and evidently

$$\lim_{(a/t)\to 0}\int_0^{a/t}\frac{a^*}{t}\left[\eta_R\left(\frac{a^*}{t}+\frac{f}{t},\frac{a^*}{t}-\frac{f}{t}\right)+\eta_L\left(\frac{a^*}{t}+\frac{f}{t},\frac{a^*}{t}-\frac{f}{t}\right)\right]\mathrm{d}\frac{a^*}{t} = 0$$

$$\lim_{(s/t)\to 0}\left(\frac{a}{t}\right)^2\left[\eta_R\left(\frac{a_1}{t},\frac{a_2}{t}\right)\pm\eta_L\left(\frac{a_1}{t},\frac{a_2}{t}\right)\right]=0 \tag{A40}$$

Then, from eqn (A38), it can be known that

$$\Delta=0 \tag{A41}$$

So,

$$F\left(\frac{a_1}{t},\frac{a_2}{t}\right)=\frac{\pi}{2m_3}\int_0^1\omega\left[\eta_R^2\left(\frac{a}{t}\omega+\frac{f}{t},\frac{a}{t}\omega-\frac{f}{t}\right)+\eta_L^2\left(\frac{a}{t}\omega+\frac{f}{t},\frac{a}{t}\omega-\frac{f}{t}\right)\right]\mathrm{d}\omega$$
$$-\left(\frac{m_1}{m_3}-1\right)\cdot 2f\left(\frac{a_1}{t},\frac{a_2}{t}\right)-\frac{n_1}{m_3}2h\left(\frac{a_1}{t},\frac{a_2}{t}\right) \tag{A42}$$

In general, the crack sliding displacements can be expressed as follows:

$$w=w_{01}\left\{n\alpha\left(\frac{a_1}{t},\frac{a_2}{t}\right)\left[\beta\left(\frac{x}{a}\right)\right]^{1/2}+\left[1-n\alpha\left(\frac{a_1}{t},\frac{a_2}{t}\right)\right]\left[\beta\left(\frac{x}{a}\right)\right]^{3/2}\right\}+w_{02}\left[\beta\left(\frac{x}{a}\right)\right]^{1/2}\frac{x}{a} \tag{A43}$$

where

$$n=2,\quad \beta\left(\frac{x}{a}\right)=1-\left(\frac{x}{a}\right)^2 \tag{A44}$$

A2. The anti-plane tearing displacement of 2-D case

A2.1. The anti-plane tearing displacement of 2-D unsymmetric inner crack

For anti-plane problems in the theory of elasticity, the displacement and stress components can be expressed as follows:

$$w=g(z)+\bar{g}(\bar{z}) \tag{A45}$$

$$\sigma_{zx}=\mu\{g'(z)+\bar{g}'(\bar{z})\},\quad \sigma_{xy}=i\mu\{g'(z)-\bar{g}'(\bar{z})\} \tag{A46}$$

For an inner crack shown in Fig. 2A(a), as the crack surface is traction-free, we have

$$g'(z)=(z^2-a^2)^{-(1/2)}\sum_{m=0}^{M}P_m z^m+\sum_{m=0}^{N}Q_n z^n \tag{A47}$$

and

$$g(z)=\sum_{m=0}^{M}P_m\gamma^m(z)+\sum_{m=0}^{N}Q_n\frac{z^n}{n+1} \tag{A48}$$

where, $\gamma_m(z)$ can be obtained by recurrence formula also.

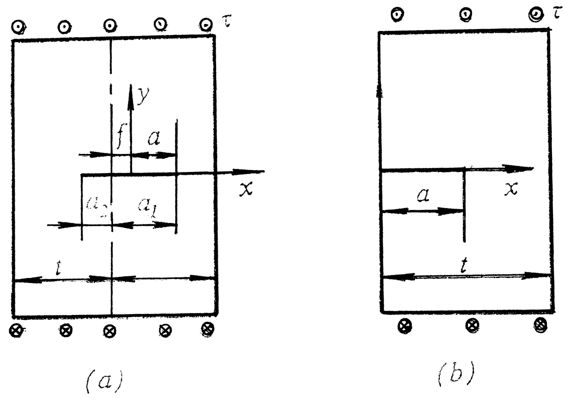

Fig. 2A. Anti-plane tearing.

Furthermore, by means of single-valued condition of displacements, the crack tearing displacement of a finite plate can be expressed by three terms as follows:

$$w = \frac{\tau_0}{2\mu} a \left\{ 2f\left(\frac{a_1}{t}, \frac{a_2}{t}\right)\left(1-\frac{x^2}{a^2}\right)^{1/2} + 2g\left(\frac{a_1}{t}, \frac{a_2}{t}\right)\left(1-\frac{x^2}{a^2}\right)^{3/2} \right\} + \frac{\tau_0}{2\mu} a 2h\left(\frac{a_1}{t}, \frac{a_2}{t}\right)\left(1-\frac{x^2}{a^2}\right)^{1/2}\frac{x}{a} \quad \text{(A49)}$$

The first kind of terms and the second kind of terms are used to denote the symmetric and antisymmetric displacements, respectively. f, q and h are unknown functions.

The generalized symmetric displacement w_{01} can be defined as the crack tearing displacement at origin and from (A49), it is equal to

$$w_{01} = \frac{\tau_0}{2\mu} a \left[2f\left(\frac{a_1}{t}, \frac{a_2}{t}\right) + 2g\left(\frac{a_1}{t}, \frac{a_2}{t}\right) \right] = \frac{\tau_0}{2\mu} a F\left(\frac{a_1}{t}, \frac{a_2}{t}\right) \quad \text{(A50)}$$

where

$$F\left(\frac{a_1}{t}, \frac{a_2}{t}\right) = 2f\left(\frac{a_1}{t}, \frac{a_2}{t}\right) + 2g\left(\frac{a_1}{t}, \frac{a_2}{t}\right) \quad \text{(A51)}$$

Furthermore, the generalized antisymmetric displacement will be

$$w_{02} = a \left.\frac{\mathrm{d}w}{\mathrm{d}(x)}\right|_{x=0} = \frac{\tau_0}{2\mu} a 2h\left(\frac{a_1}{t}, \frac{a_2}{t}\right) \quad \text{(A52)}$$

Let

$$\alpha\left(\frac{a_1}{t},\frac{a_2}{t}\right)=f\left(\frac{a_1}{t},\frac{a_2}{t}\right)\Big/F\left(\frac{a_1}{t},\frac{a_2}{t}\right) \tag{A53}$$

Substituting (A50) to (A53) into (A49) it can be obtained that

$$w = w_{01}\left\{2\alpha\left(\frac{a_1}{t},\frac{a_2}{t}\right)\left(1-\frac{x^2}{a^2}\right)^{1/2}+\left[1-2\alpha\left(\frac{a_1}{t},\frac{a_2}{t}\right)\right]\left(1-\frac{x^2}{a^2}\right)^{3/2}\right\} + w_{02}\left(1-\frac{x^2}{a^2}\right)^{1/2}\frac{x}{a} \tag{A54}$$

It is assumed that the stress intensity factors are given, and can be written as follows:

$$K_{\mathrm{III}R} = \eta_R\left(\frac{a_1}{t},\frac{a_2}{t}\right)\tau_0\sqrt{\pi a}$$

$$K_{\mathrm{III}L} = \eta_L\left(\frac{a_1}{t},\frac{a_2}{t}\right)\tau_0\sqrt{\pi a} \tag{A55}$$

Factors η_R and η_L are given functions of a_1/t and a_2/t.

From the relationship between the stress intensity factors and the crack tearing displacement in the vicinity of the crack-tip, it can be obtained that

$$f\left(\frac{a_1}{t},\frac{a_2}{t}\right)=\left[\eta_R\left(\frac{a_1}{t},\frac{a_2}{t}\right)+\eta_L\left(\frac{a_1}{t},\frac{a_2}{t}\right)\right]\Big/2$$

$$h\left(\frac{a_1}{t},\frac{a_2}{t}\right)=\left[\eta_R\left(\frac{a_1}{t},\frac{a_2}{t}\right)-\eta_L\left(\frac{a_1}{t},\frac{a_2}{t}\right)\right]\Big/2 \tag{A56}$$

Then, we have

$$\alpha\left(\frac{a_1}{t},\frac{a_2}{t}\right)=\left[\eta_R\left(\frac{a_1}{t},\frac{a_2}{t}\right)+\eta_L\left(\frac{a_1}{t},\frac{a_2}{t}\right)\right]\Big/2F\left(\frac{a_1}{t},\frac{a_2}{t}\right)$$

$$w_{01} = \frac{\tau_0}{2\mu}aF\left(\frac{a_1}{t},\frac{a_2}{t}\right)$$

$$w_{02} = w_{01}\left[\eta_R\left(\frac{a_1}{t},\frac{a_2}{t}\right)-\eta_L\left(\frac{a_1}{t},\frac{a_2}{t}\right)\right]\Big/F\left(\frac{a_1}{t},\frac{a_2}{t}\right)$$

$$\zeta = w_{02}/w_{01} = \left[\eta_R\left(\frac{a_1}{t},\frac{a_2}{t}\right)-\eta_L\left(\frac{a_1}{t},\frac{a_2}{t}\right)\right]\Big/F\left(\frac{a_1}{t},\frac{a_2}{t}\right) \tag{A57}$$

For a plate of unit thickness, it can be assumed that the virtual crack extension is a proportional one

$$\mathrm{d}a = \mathrm{d}a_1 = \mathrm{d}a_2 = a\,\mathrm{d}g \tag{A58}$$

Then, from the energy release rate method, we can obtain

$$F\left(\frac{a_1}{t},\frac{a_2}{t}\right) = \frac{\pi}{2m_3}\int_0^1 \omega\left[\eta_R^2\left(\frac{a}{t}\omega+\frac{f}{t},\frac{a}{t}\omega-\frac{f}{t}\right)+\eta_L^2\left(\frac{a}{t}\omega+\frac{f}{t},\frac{a}{t}\omega-\frac{f}{t}\right)\right]\mathrm{d}\omega - \left(\frac{m_1}{m_3}-1\right)\cdot 2f\left(\frac{a_1}{t},\frac{a_2}{t}\right)-\frac{n_1}{m_3}2h\left(\frac{a_1}{t},\frac{a_2}{t}\right) \tag{A59}$$

where

$$m_1 = \frac{1}{2}\int_{-1}^{1}\left(1-\frac{x^2}{a^2}\right)^{1/2} s(x)\,\mathrm{d}\frac{x}{a},\quad m_3 = \frac{1}{2}\int_{-1}^{1}\left(1-\frac{x^2}{a^2}\right)^{3/2} s(x)\,\mathrm{d}\frac{x}{a}$$

$$n_1 = \frac{1}{2}\int_{-1}^{1}\left(1-\frac{x^2}{a^2}\right)^{1/2} s(x)\frac{x}{a}\,\mathrm{d}\frac{x}{a} \tag{A60}$$

So the crack tearing displacement can be obtained

$$w = w_{01}\left\{n\alpha\left(\frac{a_1}{t},\frac{a_2}{t}\right)\left[\beta\left(\frac{x}{a}\right)\right]^{1/2}+\left[1-n\alpha\left(\frac{a_1}{t},\frac{a_2}{t}\right)\right]\left[\beta\left(\frac{x}{a}\right)\right]^{3/2}\right\}+w_{02}\left[\beta\left(\frac{x}{a}\right)\right]^{1/2}\frac{x}{a} \tag{A61}$$

where

$$n = 2,\quad \beta\left(\frac{x}{a}\right) = 1-\left(\frac{x}{a}\right)^2 \tag{A62}$$

A2.2. The anti-plane tearing displacement of 2-D edge crack

Now let us consider the plate with edge crack. The stress and displacement states of a plate with a Mode III edge crack can be described by means of the states of a symmetric plate with a Mode III inner crack, as shown in Fig. 2A(a). As the shear stress field is symmetric with respect to the symmetric line, as shown in Fig. 2A(a), it can be known that there is not shear stress component τ_{zx} along the symmetric line. So we can separate the plate along the symmetric line into two parts, each part can be considered as a plate with an edge crack as shown in Fig. 2A(b). The crack tearing displacement of the plate with an edge crack as shown in Fig. 2A(b) is the same with that of the plate with an inner symmetric crack.

Then, the crack tearing displacement can be expressed as follows:

$$w = w_0\left\{n\left(\frac{a}{t}\right)\left[\beta\left(\frac{x}{a}\right)\right]^{1/2}+\left[1-na\left(\frac{a}{t}\right)\right]\left[\beta\left(\frac{x}{a}\right)\right]^{3/2}\right\} \tag{A63}$$

where

$$n = 2$$

$$\beta\left(\frac{x}{a}\right) = 1-\left(\frac{x}{a}\right)^2$$

$$\alpha\left(\frac{a}{t}\right) = \eta\left(\frac{a}{t}\right) \Big/ F\left(\frac{a}{t}\right)$$

$$\eta\left(\frac{a}{t}\right) = K_{\mathrm{II}}/\tau_0\sqrt{\pi a}$$

$$F\left(\frac{a}{t}\right) = \frac{\pi}{2m_3}\int_0^1 \omega\eta^2\left(\frac{a}{t}\omega\right)\mathrm{d}\omega-\left(\frac{m_1}{m_3}-1\right)2\eta\left(\frac{a}{t}\right)$$

$$w_0 = \frac{\tau_0}{2\mu}F\left(\frac{a}{t}\right) \tag{A64}$$

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